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Characterization of Weights in Best Rational Weighted Approximation of Piecewise Smooth Functions, I

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1. INTRODUCTION

It is well known that although the collection $\mathbf{R}_n$ of all rational functions $r_n = p_n/q_n$, where p_n and q_n are in the collection π_n of all polynomials of degree n, is a much larger class than π_n, it does not improve the orders of approximation in general. For instance, the approximation order of the class Lip α, $0 < \alpha \leqslant 1$, from both $\mathbf{R}_n$ and π_n is $O(n^{-\alpha})$. So, why do we study rational approximation when it is so much easier to obtain polynomial approximants? There are at least two very good reasons. First, certain physical models are described by rational functions. An important example is the realization of a digital filter. While polynomials give only finite impulse responses, the transfer function of a digital filter described by a rational function is recursive, and with the feedback parameters, yields infinite responses. The second reason is more familiar to the approximation theorist, namely: while best approximation from π_n is saturated, this is certainly not the case in approximation by rational functions. The most

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famous example is the one given by Newman [6] where uniform approximation of $|x|$ on $[-1, 1]$ from $\mathbf{R}_n$ was considered. Although the order of approximation from π_n is only $O(n^{-1})$, it is $O(e^{-\sqrt{n}})$ from $\mathbf{R}_n$, which is a very substantial improvement. Newman's work has generated much interest in approximating piecewise smooth functions by rational functions in the late sixties and early seventies (cf. [1, 4, 5, 8, 9], for instance). In digital filter theory, the given ideal amplitude filter characteristic is also a piecewise linear function and it must be realized by means of a rational function. Judging from the previous work on rational approximation in recursive digital filter design (cf. [2, 3], for example), we believe that rational approximation with some suitable weight functions improve the filter performance. This motivates our research in characterization of weights in weighted rational approximation of piecewise smooth functions, and in particular, piecewise analytic functions.

To facilitate our discussion, we need the following notation and definitions. Let $\Delta: 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$ be a partition of the interval $[0, 1]$. For convenience, we will also use Δ to denote the set $\{x_1, ..., x_m\}$ of interior partition points. Denote by $A(\Delta)$ the collection of all complex-valued continuous functions on $[0, 1]$ whose restrictions on each $I_j = [x_j, x_{j+1}]$ are analytic on I_j, $j = 0, ..., m$, and by $C^S(\Delta)$, the collection of those whose restrictions on each I_j belong to $C^s(I_j)$, the class of functions with sth order continuous derivatives on I_j, $j = 0, ..., m$. Let w be an arbitrary weight function; $0 < w(x) < \infty$ for almost all x on $[0, 1]$. For any measurable function f defined on $[0, 1]$, we will use the notation

$$\|f\|_{L_p(w)} = \begin{cases} \{\int_0^1 |f(x)|^p \, w(x) \, dx\}^{1/p} & \text{if } 0 < p < \infty, \\ \text{ess sup}_{0 \leqslant x \leqslant 1} |f(x)| \, w(x) & \text{if } p = \infty \end{cases}$$

and

$$L_p(w) = \{f : \|f\|_{L_p(w)} < \infty\}, \qquad 0 < p \leqslant \infty.$$

Of course, if $1 \leqslant p \leqslant \infty$, $\|\cdot\|_{L_p(w)}$ defines a norm for the space $L_p(w)$. To be more precise, we let $\mathbf{R}_n[a, b]$ denote the collection of all rational functions p_n/q_n where p_n are in π_n and are relatively prime with $q_n(x) \neq 0$ for all x in $[a, b]$. In addition, set $\mathbf{R}_n = \mathbf{R}_n[0, 1]$ and $\mathbf{R} = \bigcup_n \mathbf{R}_n$. The "distance" of f from $\mathbf{R}_n$ will be denoted by

$$e_n(f)_{L_p(w)} = \inf\{\|f - r_n\|_{L_p(w)} : r_n \in \mathbf{R}_n\},$$

where $0 < p \leqslant \infty$, and for any weight function w on $[0, 1]$, set

$$U_p(w) = \left\{x \in [0, 1] : \int_{[x-\delta, x+\delta] \cap [0,1]} w(t) \, dt = \infty, \text{ for all } \delta > 0\right\}$$

if $0<p<\infty$, and

$$U_\infty(w)=\{x\in[0,1]:\ \operatorname*{ess\,sup}_{[x-\delta,x+\delta]\cap[0,1]} w(t)=\infty,\ \text{for all } \delta>0\}.$$

For any sets

$$\Theta=\{\theta_1,...,\theta_k\} \qquad \text{and} \qquad \mathcal{M}=\{\mu_1,...,\mu_k\},$$

where $0\leqslant\theta_1<\cdots<\theta_k\leqslant 1$ and $\mu_1,...,\mu_k>0$, denote by $W_p(\Theta,\mathcal{M})$, $0<p\leqslant\infty$, the collection of all weight functions w on $[0,1]$ that satisfy the following conditions:

(i) $U_p(w)=\Theta$ and

(ii) $\prod_{s=1}^k |\cdot-\theta_s|^{\mu_s}\in L_p(w)$.

For any constant $B>1$, let $\delta_n=\delta_n(B)=B^{-\sqrt{n}}$, and for any given weight function w in $W_p(\Theta,\mathcal{M})$ and a small $\delta>0$, write

$$\mathscr{E}^-_{n,s}(B)=\delta_n\|\chi_{[\theta_s-\delta,\theta_s-\delta_n]}\|_{L_p(w)}+\|(\cdot-\theta_s)\,\chi_{[\theta_s-\delta_n,\theta_s]}(\cdot)\|_{L_p(w)},$$

$$\mathscr{E}^+_{n,s}(B)=\delta_n\|\chi_{[\theta_s+\delta_n,\theta_s+\delta]}\|_{L_p(w)}+\|(\cdot-\theta_s)\,\chi_{[\theta_s,\theta_s+\delta_n]}(\cdot)\|_{L_p(w)},$$

and

$$\mathscr{E}_n(B)=\sum_{\theta_s\in\Theta\cap\Delta}\min(\mathscr{E}^-_{n,s}(B),\mathscr{E}^+_{n,s}(B)),$$

where, and throughout, as usual, χ_J denotes the characteristic function of the set J and an empty sum is considered to be zero.

Our main result in this paper is the following:

THEOREM 1. *Let $0<p\leqslant\infty$. Then a necessary and sufficient condition for $e_n(f)_{L_p(w)}\to 0$, as $n\to\infty$, where f is an arbitrary function in $A(\Delta)\backslash\mathbf{R}$, is that there exist Θ and $\mathcal{M}$ such that $w\in W_p(\Theta,\mathcal{M})$ and if $\Theta\cap\Delta\neq\phi$, then to each $\theta_s\in\Theta\cap\Delta$, it follows that*

$$\lim_{\delta\to 0^+}\|(\cdot-\theta_s)\,\chi_{[\theta_s-\delta,\theta_s]}(\cdot)\|_{L_p(w)}=0 \tag{α}$$

or

$$\lim_{\delta\to 0^+}\|(\cdot-\theta_s)\,\chi_{[\theta_s,\theta_s+\delta]}(\cdot)\|_{L_p(w)}=0. \tag{β}$$

Furthermore, if $w \in W_p(\Theta, \mathcal{M})$ *for some* Θ *and* $\mathcal{M}$, *then there exist constants* A *and* B, *with both* $A > 1$ *and* $B > 1$, *such that*

$$e_n(f)_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}_n(B)) \tag{1}$$

for any f *in* $A(\Delta)$.

It is well known (cf. [10, 11]) that if $w \equiv 1$, then $e_n(f)_{L_p(1)} = O(e^{-\lambda\sqrt{n}})$ for any f in $A(\Delta)$. Hence, it would be of some interest to characterize the weight functions w for which

$$e_n(f)_{L_p(w)} = O(e^{-\lambda\sqrt{n}}) \tag{2}$$

for some $\lambda > 0$ and any f in $A(\Delta)$. Our result in this direction can be stated as follows.

THEOREM 2. *If corresponding to every* $\theta_s \in \Theta \cap \Delta$, *we have* $\mu_s < 1$, *then there is a* $\lambda > 0$ *such that for any* $0 < p \leqslant \infty$ *and any* $w \in W_p(\Theta, \mathcal{M})$, *the estimate in* (2) *holds for all* f *in* $A(\Delta)$.

On the other hand, if there is a $\theta_{s0} \in \Theta \cap \Delta$ *such that the corresponding* μ_{s0} *is at least* 1, *then to any positive* λ, *there exists a* w *in* $W_\infty(\Theta, \mathcal{M})$, *satisfying* (α) *and* (β), *and an* $f \in A(\Delta)$ *so that the sequence* $\{e^{\lambda\sqrt{n}} e_n(f)_{L_\infty(w)}\}$ *is unbounded.*

2. PRELIMINARY RESULTS

We need several lemmas. The first one is a result of Newman [6].

LEMMA 1. *Let* $\eta = \exp(-1/\sqrt{n})$ *and* $p(x) = \prod_{k=0}^{n-1} (x + \eta^k)$. *Then*

$$\left|\frac{p(-x)}{p(x)}\right| \leqslant \eta^n$$

for $\eta^n \leqslant x \leqslant 1$.

The second result we need is the following.

LEMMA 2. *Let* $\xi_1, ..., \xi_q \in [-1, 0) \cup (0, 1]$, $\mu > 0$ *and* $\mu_j > 0$, $j = 1, ..., q$. *Then for any constants* δ, B, C, ε, *and* $\varepsilon_1, ..., \varepsilon_q$ *satisfying*

$$0 < \delta < \tfrac{1}{2}, \qquad 1 < B^{[\mu]+1} < e, \qquad C > 1, \qquad \text{and} \quad e > 0,$$

there exist rational functions $r_n \in \mathbf{R}_{m_n}[-1, 1]$, *where* $m_n = n + O(\sqrt{n})$, *such that*

$$|\operatorname{sgn} x - r_n(x)| = \begin{cases} O(1) & \text{for } x \in [-\eta^n, \eta^n], \\ O\left(\left(\frac{B^{[\mu]+1}}{e}\right)^{\sqrt{n}}\right) \prod_{\xi_j > 0} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} \, |x - \varepsilon B^{-\sqrt{n}}|^{\mu} & \\ \qquad \text{for } x \in [\eta^n, 1], & \\ O\left(\left(\frac{B^{[\mu]+1}}{e}\right)^{\sqrt{n}}\right) \prod_{\xi_j < 0} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} \, |x - \varepsilon B^{-\sqrt{n}}|^{\mu} & \\ \qquad \text{for } x \in [-1, -\eta^n], & \\ O(C^{-\sqrt{n}}) \prod_{j=1}^{q} |x - \xi_j - \varepsilon_j B^{-\sqrt{n}}|^{\mu_j} & \\ \qquad \text{for } \delta \leqslant |x| \leqslant 1, & \end{cases}$$

where $\eta = e^{-n^{-1/2}}$ *and the "O" terms are independent of* x.

Proof. We define our r_n by

$$r_n(x) = \frac{P_1(x)\, P_2(x)\, P_3(x) - P_1(-x)\, P_2(-x)\, P_3(-x)}{P_1(x)\, P_2(x)\, P_3(x) + P_1(-x)\, P_2(-x)\, P_3(-x)},$$

where

$$P_1(x) = \prod_{0 \leqslant k \leqslant (2\ln(1/\delta)+1)\sqrt{n}} (x + \eta^k)^{1 + [\ln C/(\ln \delta)(\ln(1-\delta))]},$$

$$P_2(x) = \prod_{(2\ln(1/\delta)+1)\sqrt{n} < k < n} (x + \eta^k),$$

and

$$P_3(x) = \prod_{j=1}^{q} (x + |\xi_j + \varepsilon_j B^{-\sqrt{n}}|)^{[\mu_j]+1} \, (x + \varepsilon B^{-\sqrt{n}})^{[\mu]+1}.$$

It is clear that $r_n \in \mathbf{R}_{m_n}$ where $m_n = n + O(\sqrt{n})$. To verify the other required properties, we note that since both $\operatorname{sgn}(x)$ and $r_n(x)$ are odd functions of x, it is sufficient to consider $0 \leqslant x \leqslant 1$. In the following estimates, n is always assumed to be sufficiently large.

For $0 \leqslant x \leqslant \eta^n$, since both $[P_1(x)\, P_2(x)\, P_3(x)]$ and $[P_1(-x)\, P_2(-x) \times P_3(-x)]$ are positive, we actually have $|r_n(x)| < 1$.

Next, let $\eta^n \leqslant x \leqslant 1$. Then

$$
\begin{aligned}
|\operatorname{sgn} x - r_n(x)| &= \frac{2|P_1(-x)\,P_2(-x)\,P_3(-x)|}{|P_1(x)\,P_2(x)\,P_3(x) + P_1(-x)\,P_2(-x)\,P_3(-x)|} \\
&\leqslant \frac{2\,|P_3(-x)|}{\Phi(x) - |P_3(-x)|} \\
&\leqslant \frac{O(\prod_{\xi_j>0} |x-\xi_j-\varepsilon_j B^{-\sqrt{n}}|^{\mu_j}\,|x-\varepsilon B^{-\sqrt{n}}|^{\mu})}{\Phi(x) - K_n},
\end{aligned} \tag{3}
$$

where $K_n = \max\{|P_3(x)| : 0 \leqslant x \leqslant 1\}$ and

$$
\Phi(x) = \left|\frac{P_1(x)\,P_2(x)\,P_3(x)}{P_1(-x)\,P_2(-x)}\right|.
$$

Since $\eta^n \leqslant x \leqslant 1$, we have, from Lemma 1, that

$$
\begin{aligned}
\frac{1}{\Phi(x)} &\leqslant \prod_{j=0}^{n-1} \left|\frac{x-\eta^j}{x+\eta^j}\right| \frac{1}{(\varepsilon B^{-\sqrt{n}})^{[\mu]+1} \prod_{j=1}^{q} |\xi_j + \varepsilon_j B^{-\sqrt{n}}|^{[\mu_j]+1}} \\
&\leqslant \frac{1}{\varepsilon^{[\mu]+1} \prod_{j=1}^{q} |(1/2)\,\xi_j|^{[\mu_j]+1}} (B^{[\mu]+1}/e)^{\sqrt{n}},
\end{aligned} \tag{4}
$$

so that the estimate for $x \in [\eta^n, 1]$ is established.

Finally, let $\delta \leqslant x \leqslant 1$. Then k_0 can be chosen, depending on x, such that $\eta^{k_0+1} \leqslant x \leqslant \eta^{k_0}$. But independent of x, we have $k_0 \leqslant [\sqrt{n}\ln(1/\delta)]$. Thus, it follows that

$$
\begin{aligned}
\frac{1}{\Phi(x)} &= O(1) \left|\frac{P_1(-x)}{P_1(x)}\right| \\
&= O(1) \prod_{k_0+1 \leqslant k \leqslant (2\ln(1/\delta)+1)\sqrt{n}} \left|\frac{x-\eta^k}{x+\eta^k}\right|^{[\ln C/(\ln\delta)(\ln(1-\delta))]+1} \\
&= O(1) \prod_{k_0+1 \leqslant k \leqslant (2\ln(1/\delta)+1)\sqrt{n}} |x-\eta^k|^{[\ln C/(\ln\delta)(\ln(1-\delta))]+1} \\
&= O(1)(1-\delta)^{\sqrt{n}\ln(1/\delta)\{[\ln C/(\ln\delta)(\ln(1-\delta))]+1\}} \\
&= O(C^{-\sqrt{n}}).
\end{aligned}
$$

Hence, by using (3) and (4), we have verified the estimate for $\delta \leqslant x \leqslant 1$.

Remark 1. Lemma 2 holds if $B = 1$ and $\xi_j + \varepsilon_j \neq 0$, $j = 1, ..., q$.

The following result of Bernstein is well known.

LEMMA 3. *Let f be analytic on $[a, b]$. Then there exists a sequence of polynomials p_n in π_n such that*

$$\|f - p_n\|_{L_\infty[a,b]} = O(e^{-\lambda n})$$

for some $\lambda > 0$.

The key lemma in this paper is the following result.

LEMMA 4. *Let $\Theta = \{\theta_1, ..., \theta_k\}$, $\mathcal{M} = \{\mu_1, ..., \mu_k\}$, and $\Delta = \{x_1, x_2\}$ be given. If $w \in W_p(\Theta, \mathcal{M})$ and*

$$f(x) = \chi_{[x_1, x_2]}(x)(x - x_1)(x - x_2) f_1(x),$$

where f_1 is analytic on $[x_1, x_2]$, then

$$e_n(f)_{L_p(w)} = O(A_1^{-\sqrt{n}}) + O(\varepsilon_n(B_1)) \tag{5}$$

for some constants $A_1 > 1$ and $B_1 > 1$.

Proof. Choose a sufficiently small $\delta > 0$ so that f_1 is analytic on $[x_1 - \delta, x_2 + \delta]$ and $\theta_s \notin [x_1 - \delta, x_1) \cup (x_2, x_2 + \delta]$, $s = 1, ..., k$. Construct a polynomial p_0 of degree $\leqslant \sum_{\theta_s \in [x_1, x_2]}([\mu_s] + 1)$ such that

$$p_0(x) - f_1(x) = \prod_{\theta_s \in [x_1, x_2]} (x - \theta_s)^{[\mu_s] + 1} g(x),$$

where g is also analytic on $[x_1 - \delta, x_2 + \delta]$. By Lemma 3, there is a polynomial p_1 of degree $K[\sqrt{n}] - \sum_s([\mu_s] + 1)$ such that

$$|g(x) - p_1(x)| = O(e^{-\sqrt{n}})$$

uniformly on $[x_1 - \delta, x_2 + \delta]$. Set

$$p_2(x) = p_0(x) - \prod_{\theta_s \in [x_1, x_2]} (x - \theta_s)^{[\mu_s] + 1} p_1(x).$$

Then p_2 is a polynomial of degree $K[\sqrt{n}]$ and satisfies, uniformly on $[x_1 - \delta, x_2 + \delta]$,

$$p_2(x) - f_1(x) = O(e^{-\sqrt{n}}) \prod_{\theta_s \in [x_1, x_2]} |x - \theta_s|^{\mu_s}. \tag{6}$$

First, let us assume that $\Theta \cap \Delta = \phi$. Then by Remark 1, we see that there exists a rational function r_n of degree $2n + O(\sqrt{n})$ such that

$$|r_n(x)-\chi_{[x_1,x_2]}| = \begin{cases} O(1) \quad \text{for} \quad x\in[x_1-\eta^n, x_1+\eta^n]\cup[x_2-\eta^n, x_2+\eta^n], \\ O(e^{-\sqrt{n}}) \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} \\ \qquad \text{for} \quad x\notin[x_1-\eta^n, x_1+\eta^n]\cup[x_2-\eta^n, x_2+\eta^n], \\ O(C^{-\sqrt{n}}) \prod_{s=1}^{n} |x-\theta_s|^{\mu_s} \\ \qquad \text{for} \quad x\in[x_1-\delta, x_2+\delta]. \end{cases}$$

Since there is some $\lambda'>0$ such that

$$p_2(x)=O(e^{\lambda'\sqrt{n}})$$

uniformly on $[0, 1]$, we have, by setting $C=e^{\lambda'+1}$,

$$\begin{aligned} f(x) &= \chi_{[x_1,x_2]}(x)(x-x_1)(x-x_2)(f_1(x)-p_2(x)) \\ &\quad + \chi_{[x_1,x_2]}(x)(x-x_1)(x-x_2)\, p_2(x) \\ &= O(e^{-\sqrt{n}}) \prod_s |x-\theta_s|^{\mu_s} + r_n(x)(x-x_1)(x-x_2)\, p_2(x). \end{aligned}$$

This implies that

$$\begin{aligned} e_{m_n}(f)_{L_p(w)} &\leqslant \|f(\cdot)-r_n(\cdot)(\cdot-x_1)(\cdot-x_2)\, p_2(\cdot)\|_{L_p(w)} \\ &= O(e^{-\sqrt{n}}) \left\| \prod_s |\cdot-\theta_s|^{\mu_s} \right\|_{L_p(w)} = O(e^{-\sqrt{n}}). \end{aligned}$$

Also, since $m_n = 2n + O(\sqrt{n})$, we have

$$e_n(f)_{L_p(w)} = O(e^{-\sqrt{n/2}})\, O(K^{n^{1/4}}) = O(e^{-\sqrt{n/3}}).$$

Now suppose that $x_1=\theta_{s_0}\in\Theta$. Set $x'=x_1-2B^{-\sqrt{n}}$, where

$$B=\frac{1}{2}\left\{1+\exp\left(\frac{1}{[\mu_{s_0}]+1}\right)\right\}.$$

If n is large enough, then $\theta_s\notin[x', x_1)$, $s=1, ..., k$, and an application of (6) yields

$$\begin{aligned} f(x) &= O(e^{\sqrt{n}}) \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} + \chi_{[x',x_2]}(x)(x-x_1)(x-x_2)\, p_2(x) \\ &\quad - \chi_{[x',x_1]}(x)(x-x_1)(x-x_2)\, p_2(x) \\ &:= O(e^{-\sqrt{n}}) \prod_{s=1}^{k} |x-\theta_s|^{\mu_s} + J_1 - J_2. \end{aligned} \tag{7}$$

It is easy to see that

$$\|J_2\|_{L_p(w)} \leqslant C \| |\cdot - x_1| \chi_{[x_1 - 2B^{-\sqrt{n}}, x_1 - B^{-\sqrt{n}}]}(\cdot)\|_{L_p(w)} + C \| |\cdot - x_1| \chi_{[x_1 - B^{-\sqrt{n}}, x_1]}(\cdot)\|_{L_p(w)} \leqslant O(1)\, \mathscr{E}^-_{n,s_0}(B). \quad (8)$$

By Lemma 2, there exists a rational function r_n of degree $2n + O(\sqrt{n})$ such that

$$|\chi_{[x', x_2]}(x) - r_n(x)| = \begin{cases} O(1) & \text{for } x \in [x' - \eta^n, x' + \eta^n] \cup [x_2 - \eta^n, x_2 + \eta^n], \\ O(B^{[\mu_{s_0}]+1}/e)^{\sqrt{n}} \prod_{s=1}^{k} |x - \theta_s|^{\mu_s} & \text{for } x \in [x' + \eta^n, x_2 - \eta^n] \cup [0, x' - \eta^n] \cup [x_2 + \eta^n, 1], \\ O(C^{-\sqrt{n}}) \prod_{s=1}^{k} |x - \theta_s|^{\mu_s} & \text{for } x \in [0, 1] \backslash [x_1 - \delta, x_2 + \delta]. \end{cases}$$

Thus, if we write

$$\begin{aligned} J_1 &= (\chi_{[x', x_2]}(x) - r_n(x))(x - x_1)(x - x_2)\, p_2(x) \\ &\quad + (x - x_1)(x - x_2)\, p_2(x)\, r_n(x) \\ &:= J_1' + r_n^*(x), \end{aligned} \quad (9)$$

then

$$\begin{aligned} \|J_1 \chi_{[x' - \eta^n, x' + \eta^n]}\|_{L_p(w)} &= O(1) \|(\cdot - x_1)\, \chi_{[x' - \eta^n, x' + \eta^n]}(\cdot)\|_{L_p(w)} \\ &= O(B^{-\sqrt{n}} \|\chi_{[x_1 - \delta, x_1 - B^{-\sqrt{n}}]}\|_{L_p(w)}) = O(\mathscr{E}^-_{n,s_0}(B)). \end{aligned}$$

By setting $C = \exp(\lambda' + 1)$, it follows from (7), (8), and (9), that

$$\|f - r_n^*\|_{L_p(w)} = O(e^{-\sqrt{n}}) + O(B^{[\mu_{s_0}]+1}/e)^{\sqrt{n}} + O(\mathscr{E}^-_{n,s_0}(B)),$$

where r_n^* is a rational function of degree $2n + O(\sqrt{n})$. By setting

$$A_1 = (e/B^{[\mu_{s_0}]+1})^{1/\sqrt{3}} \qquad \text{and} \qquad B_1 = B^{1/\sqrt{3}},$$

we obtain

$$e_n(f)_{L_p(w)} = O(A_1^{-\sqrt{n}}) + O(\mathscr{E}^-_{n,s_0}(B_1)).$$

Similarly, replacing $\mathscr{E}^-_{n,s_0}(B_1)$ by $\mathscr{E}^+_{n,s_0}(B_1)$, we also obtain

$$e_n(f)_{L_p(w)} = O(A_1^{-\sqrt{n}}) + O(\mathscr{E}^+_{n,s_0}(B_1)).$$

Thus we have established the lemma for the special case $\Theta \cap \Delta = \phi$.

If $x_2=\theta_{s_0}\in\Theta\cap\Delta$, then a similar estimate also gives (5).
Now, suppose that both $x_1=\theta_{s_1}$ and $x_2=\theta_{s_2}$ belong to the set Θ. Set

$$x'=x_1-2B^{-\sqrt{n}} \qquad \text{and} \qquad x''=x_2-2B^{-\sqrt{n}},$$

where

$$B=\frac{1}{2}+\frac{1}{2}\min\left\{\exp\left(\frac{1}{[\mu_{s_1}]+1}\right), \exp\left(\frac{1}{[\mu_{s_2}]+1}\right)\right\}.$$

For all sufficiently large n, we have $\theta_s\notin[x',x_1)\cup[x'',x_2)$, $s=1, ..., k$. By (6), we see that

$$\begin{aligned} f(x)&=O(e^{-\sqrt{n}})\prod_{s=1}^{k}|x-\theta_s|^{\mu_s}+\chi_{[x',x'']}(x)(x-x_1)(x-x_2)\,p_2(x)\\ &\quad-\chi_{[x',x_1]}(x)(x-x_1)(x-x_2)\,p_2(x)+\chi_{[x'',x_2]}(x)(x-x_1)(x-x_2)\,p_2(x)\\ &:=O(e^{-\sqrt{n}})\prod_{s=1}^{k}|x-\theta_s|^{\mu_s}+K_1-K_2-K_3. \end{aligned} \tag{10}$$

Therefore, we have

$$\|K_2\|_{L_p(w)}=O(1)\,\mathscr{E}^{-}_{n,s_1}(B) \tag{11}$$

and

$$\|K_3\|_{L_p(w)}=O(1)\,\mathscr{E}^{-}_{n,s_2}(B). \tag{12}$$

Write

$$\chi_{[x',x'']}(x)=\tfrac{1}{2}\{\operatorname{sgn}(x-x')-\operatorname{sgn}(x-x'')\}.$$

By Lemma 2, there are rational functions $\tilde{r}_n$ and $\hat{r}_n$ of degree $n+O(\sqrt{n})$ such that

$$|\tilde{r}_n(x)-\operatorname{sgn}(x-x')|=\begin{cases} O(1) & \text{for } |x-x'|\leqslant\eta^n,\\ O(B^{[\mu_{s_1}]+1}/e)^{\sqrt{n}}\prod_{s=1}^{k}|x-\theta_s|^{\mu_s} & \\ \qquad \text{for } |x-x'|\geqslant\eta^n \text{ and } x\in[0,1], & \\ O(C^{-\sqrt{n}})\prod_{s=1}^{k}|x-\theta_s|^{\mu_s} & \\ \qquad \text{for } |x-x'|\geqslant\delta \text{ and } x\in[0,1], & \end{cases}$$

and

$$|\hat{r}_n(x) - \operatorname{sgn}(x - x'')| = \begin{cases} O(1) & \text{for } |x - x''| \leqslant \eta^n, \\ O(B^{[\mu_{s2}]+1}/e)^{\sqrt{n}} \prod_{s=1}^{k} |x - \theta_s|^{\mu_s} & \\ & \text{for } |x - x''| \geqslant \eta^n \text{ and } x \in [0, 1], \\ O(C^{-\sqrt{n}}) \prod_{s=1}^{k} |x - \theta_s|^{\mu_s} & \\ & \text{for } |x - x''| \geqslant \delta \text{ and } x \in [0, 1]. \end{cases}$$

Set

$$r_n^*(x) = \tfrac{1}{2}(\tilde{r}_n(x) - \hat{r}_n(x))(x - x_1)(x - x_2)\, p_2(x).$$

Then by (10), (11), and (12), we have

$$\|f - r_n^*\|_{L_p(w)} = O(e^{-\sqrt{n}}) + O(B^{[\mu_{s1}]+1}/e)^{\sqrt{n}} + O(B^{[\mu_{s2}]+1}/e)^{\sqrt{n}} + O(\mathscr{E}_{n,s_1}^{-}(B)) + O(\mathscr{E}_{n,s_2}^{-}(B)),$$

and this, in turn, yields

$$e_n(f)_{L_p(w)} = O(A_1^{-\sqrt{n}}) + O(\mathscr{E}_{n,s_1}^{-}(B_1) + \mathscr{E}_{n,s_2}^{-}(B_1)),$$

for some $A_1 > 1$ and $B_1 > 1$. Similarly, we have

$$e_n(f)_{L_p(w)} = O(A_1^{-\sqrt{n}}) + O(\mathscr{E}_{n,s_1}^{-}(B_1) + \mathscr{E}_{n,s_2}^{+}(B_1)),$$

$$e_n(f)_{L_p(w)} = O(A_1^{-\sqrt{n}}) + O(\mathscr{E}_{n,s_1}^{+}(B_1) + \mathscr{E}_{n,s_2}^{+}(B_1)),$$

and

$$e_n(f)_{L_p(w)} = O(A_1^{-\sqrt{n}}) + O(\mathscr{E}_{n,s_1}^{+}(B_1) + \mathscr{E}_{n,s_2}^{-}(B_1)).$$

Hence, combining these estimates, we obtain (5). This completes the proof of Lemma 4.

Remark 2. If $x_1 = 0$ or $x_2 = 1$, then the conclusion in Lemma 4 also holds.

We also need the following lemma.

LEMMA 5. *If* $0 < \delta < \lambda$, *then*

$$\inf_{r_n \in R_n} \|\operatorname{sgn} x - r_n(x)\|_{L_\infty[-\lambda, -\delta] \cup [\delta, \lambda]} \geqslant \exp\left(-\pi^2 n / 2 \ln \frac{\lambda}{\delta}\right). \tag{13}$$

This result is a simple consequence of the following estimation derived by Gončar [5]:

$$\inf_{r_n \in R_n} \|\operatorname{sgn} x - r_n(x)\|_{L_\infty[-1,-\delta] \cup [\delta,1]} \geqslant \exp\left(-\pi^2 n/2 \ln \frac{1}{\delta}\right).$$

3. Proof of Theorem 1

We are now ready to prove the first theorem. To prove the necessity direction, let $r_n \in \mathbf{R}$ such that

$$\|f - r_n\|_{L_p(w)} \to 0. \tag{14}$$

If $U_p(w)$ were an infinite set, then we always have

$$\|e^x - r_n(x)\|_{L_p(w)} = \infty$$

for any r_n, which contradicts with (14). Let $U_p(w) = \Theta = \{\theta_1, ..., \theta_k\}$ and set $\delta = \frac{1}{4}\min_{1 \leqslant j \leqslant m-1} |\theta_j - \theta_{j+1}|$. We first observe that for every s, $s = 1, ..., k$, there is a positive μ_s such that $|\cdot - \theta_s|^{\mu_s} \chi_{[\theta_s - \delta, \theta_s + \delta] \cap [0,1]}(\cdot) \in L_p(w)$. Indeed, if for some s_0, $1 \leqslant s_0 \leqslant k$, we have

$$|\cdot - \theta_{s_0}|^M \chi_{[\theta_{s_0} - \delta, \theta_{s_0} + \delta] \cap [0,1]}(\cdot) \notin L_p(w), \qquad M = 1, 2, ...,$$

and for any $f \in A(\Delta) \backslash \mathbf{R}$ we have $e_n(f)_{L_p(w)} \to 0$, then there exist $r_n \in \mathbf{R}$ such that

$$\|(f - r_n)\chi_{[\theta_{s_0} - \delta, \theta_{s_0} + \delta] \cap [0,1]}\|_{L_p(w)} \leqslant \|f - r_n\|_{L_p(w)} \to 0$$

so that, since M and δ are arbitrary, we see that

$$f(\theta_{s_0}) = r_n(\theta_{s_0}), \qquad f'(\theta_{s_0}) = r_n'(\theta_{s_0}), ..., f^{(j)}(\theta_{s_0}) = r_n^{(j)}(\theta_{s_0}), ...,$$

and this implies that $f = r_n$ on some fixed interval. As a consequence, $r_m = r_n$ for all m, and this yields $f \equiv r_n \in \mathbf{R}$, which is a contradiction.

Suppose that for $x_j = \theta_{s_0} \in \Delta \cap \Theta$ both (α) and (β) do not hold. We first consider the case $0 < p < \infty$. Let $f(\cdot) = |\cdot - \theta_{s_0}|$. If for some sequence $\{r_n\} \subset \mathbf{R}$, we have $\|r_n - f\|_{L_p(w)} \to 0$, then r_n must be of the form

$$r_n(x) = (x - \theta_{s_0}) \frac{p_{n-1}(x)}{q_n(x)},$$

where p_{n-1} and q_n are polynomials of degrees $n-1$ and n, respectively.

Thus, for any small $\delta > 0$,

$$\int_{\theta_{s0}-\delta}^{\theta_{s0}+\delta} |r_n(x) - f(x)|^p \, w(x) \, dx$$

$$= \int_{\theta_{s0}-\delta}^{\theta_{s0}+\delta} \left| \operatorname{sgn}(x-\theta_{s0}) - \frac{p_{n-1}(x)}{q_n(x)} \right|^p |x-\theta_{s0}|^p \, w(x) \, dx < \infty. \quad (15)$$

Since both (α) and (β) do not hold, we have

$$\int_{\theta_{s0}-\delta}^{\theta_{s0}} |x-\theta_{s0}|^p \, w(x) \, dx = \infty \quad \text{and} \quad \int_{\theta_{s0}}^{\theta_{s0}+\delta} |x-\theta_{s0}|^p \, w(x) \, dx = \infty. \quad (16)$$

From (15) and (16), it follows that there are two sequences $\alpha_\ell \downarrow \theta_{s0}$ and $\beta_\ell \uparrow \theta_{s0}$ such that

$$\frac{(p_{n-1}(\alpha_\ell)/q_n(\alpha_\ell)) - (p_{n-1}(\beta_\ell)/q_n(\beta_\ell))}{\beta_\ell - \alpha_\ell} \geqslant \frac{1}{\beta_\ell - \alpha_\ell} \to \infty. \quad (17)$$

This implies that

$$\left(\frac{d}{dx} \frac{p_{n-1}(x)}{q_n(x)} \right)_{x=\theta_s} = \infty \quad (18)$$

which is not possible. Hence, for $0 < p < \infty$, (α) or (β) must hold.

Next consider $p = \infty$ and again assume that both (α) and (β) do not hold. Then there exists a positive number a such that for any $\lambda > 0$ the measures of the sets $E_\lambda^- = \{x \in [\theta_{s0} - \lambda, \theta_{s0}]: |x-\theta_{s0}| \, w(x) > a\}$ and $E_\lambda^+ = \{x \in [\theta_{s0}, \theta_{s0} + \lambda]: |x-\theta_{s0}| \, w(x) > a\}$ are positive. If $e_n(f)_{L_p(w)} \to 0$, then there exist p_{n-1} and q_n such that for all large n,

$$\operatorname*{ess\,sup}_{x \in E_\lambda^+ \cup E_\lambda^-} \left| \frac{p_{n-1}(x)}{q_n(x)} - \operatorname{sgn}(x-\theta_{s0}) \right| |x-\theta_{s0}| \, w(x) < \frac{a}{2}.$$

Hence, there exist two sequences of numbers $\alpha_\ell \downarrow \theta_{s0}$ and $\beta_\ell \uparrow \theta_{s0}$ such that (18) holds, and this is again impossible.

We now consider the sufficiency direction. Since $\mathscr{E}_n(B) \to 0$ when one of the conditions (α) or (β) is satisfied, the sufficiency direction follows from the second part of the theorem, which we are going to establish. That is, we must show that there exist $A > 1$ and $B > 1$, such that

$$e_n(f)_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}_n(B)).$$

For every $f \in A(\Delta)$ there is a polynomial p_0 of degree m such that

$$f(x) - p_0(x) = \sum_{j=0}^{m} \chi_{I_j}(x)(x - x_j)(x - x_{j+1})\, g_j(x)$$
$$:= \sum_{j=0}^{m} f_j(x),$$

where g_j is analytic on I_j, $j = 0, ..., m$. By Lemma 4 and Remark 2, we see that there exist constants $A_j > 1$ and $B_j > 1$, such that

$$e_n(f_j)_{L_p(w)} = O(A_j^{-\sqrt{n}}) + O(\mathscr{E}_n(B_j)), \qquad j = 1, ..., m.$$

Setting $A = \min_j A_j$ and $B = \min_j B_j$ completes the proof of the theorem.

4. Proof of Theorem 2

If corresponding to each $\theta_s \in \Theta \cap \Delta$ we have $\mu_s < 1$, then

$$\mathscr{E}^{\pm}_{n,s}(B) = O(B_s^{-\sqrt{n}})$$

for some $B_s > 1$. Hence, by (1) we obtain

$$e_n(f)_{L_p(w)} = O(A^{-\sqrt{n}}) + O(\mathscr{E}_n(B)) = O(e^{-\lambda\sqrt{n}}),$$

where $\lambda = \min_{\theta_s \in \Theta \cap \Delta} \{\ln A, \ln B_s\}$. This completes the proof of the first half of the theorem.

To verify the second half of the theorem, let us consider the case where $\Delta = \Theta = \{\frac{1}{2}\}$ and $\mathscr{M} = \{\lambda_0\}$ with $\lambda_0 \geqslant 1$. Let $\lambda > 0$ be arbitrarily given and we now construct our weight function w as follows.

Let $\varepsilon_n \geqslant e^{-\lambda\sqrt{n}}$ such that $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \cdots$ and $\varepsilon_n \to 0$, and define $\delta_j = \delta_{j-1}/e^{j/2}$, $j = 1, 2, ...,$ with $\delta_0 = \frac{1}{2}$. Set

$$\varepsilon_j^* = \max\left\{\varepsilon_j, \frac{1}{|\ln \delta_{j-1}|}\right\}$$

and

$$H_j = \left[\frac{1}{2} - \delta_j, \frac{1}{2} - \delta_{j+1}\right] \cup \left[\frac{1}{2} + \delta_{j+1}, \frac{1}{2} + \delta_j\right].$$

Then our weight function w is defined by

$$w(x) = \sum \sqrt{\varepsilon_{j+1}^*}\, \chi_{H_j}(x) \left|x - \frac{1}{2}\right|^{-1}.$$

It is clear that $w \in W_\infty(\Theta, \mathcal{M})$. Let

$$f(x) = \left| x - \frac{1}{2} \right|.$$

Then if

$$\|(r_{j+1} - f)w\|_{L_\infty[0,1]} < \infty,$$

r_{j+1} must be of the form

$$r_{j+1}(x) = \left(x - \frac{1}{2}\right) \frac{p_j(x)}{q_{j+1}(x)},$$

so that Lemma 5 gives

$$\begin{aligned}\|(r_{j+1} - f)w\|_{L_\infty[0,1]} &\geqslant \sqrt{\varepsilon_{j+1}^*} \left\| \frac{p_j}{q_{j+1}} - \operatorname{sgn}\left(\cdot - \frac{1}{2}\right) \right\|_{L_\infty(H_j)} \\ &\geqslant \sqrt{\varepsilon_{j+1}^*} \exp(-\pi^2(j+1)/2 \ln(\delta_j/\delta_{j+1})) \\ &= \sqrt{\varepsilon_{j+1}^*}\, e^{-\pi^2}.\end{aligned}$$

Hence, we have

$$e^{\lambda\sqrt{n}} e_n(f)_{L_\infty(w)} \geqslant \frac{1}{\varepsilon_n} e_n(f)_{L_\infty(w)} \geqslant \frac{1}{\varepsilon_n^*} e_n(f)_{L_\infty(w)} \geqslant \frac{1}{2e^{\pi^2}\sqrt{\varepsilon_n^*}} \to \infty.$$

This completes the proof of the theorem.

5. Approximation of Piecewise Smooth Functions

By modifying the proofs and constructions in the above discussions, we can also establish analogous results for the class $C^s(\Delta)$. We state these results without giving the details in the following.

THEOREM 3. *Let $0 < p \leqslant \infty$ and s be a positive integer. Then for any f in $C^s(\Delta)\backslash\mathbf{R}$, a necessary and sufficient condition for*

$$e_n(f)_{L_p(w)} \to 0$$

is that the conditions of Theorem 1 are satisfied and $\mu_j \leqslant s$ for all $j = 1, ..., k$.

Furthermore, if $w \in W_p(\Theta, \mathcal{M})$ for some Θ and $\mathcal{M}$ with $\mu_j \leqslant s$, $j = 1, ..., k$, then there exists a constant $B > 1$ such that

$$e_n(f)_{L_p(w)} = O(\mathscr{E}_n(B)) + O\left(\frac{1}{n^s} \sum_{j=0}^{m} \omega\left(f_j, \frac{1}{n}\right)_p\right),$$

where f_j denotes the restriction of f on I_j and $\omega|(f_j, 1/n)_p$ the L_p-modulus of continuity of f_j.

THEOREM 4. *Let $0 < p \leqslant \infty$ and s be a positive integer. Suppose that $w \in W_p(\Theta, \mathcal{M})$ where $\mu_j \leqslant s$ for all $j = 1, ..., k$. If to every $\theta_j \in \Theta \cap \Delta$, the corresponding μ_j is less than 1, then*

$$e_n(f)_{L_p(w)} = O\left(\frac{1}{n^s} \sum_{j=0}^{m} \omega\left(f_j, \frac{1}{n}\right)_p\right)$$

for each $f \in C^S(\Delta)$.

If there is some $\theta_{j_0} \in \Theta \cap \Delta$ with the corresponding $\mu_{j_0} \geqslant 1$, then for any arbitrary sequence $\varepsilon_n \downarrow 0$, there exists a weight function $w \in W_\infty(\Theta, \mathcal{M})$, satisfying (α) and (β) whenever $\Theta \cap \Delta \neq \phi$, and an $f \in C^s(\Delta)$ such that

$$\frac{1}{\varepsilon_n} e_n(f)_{L_p(w)} \to \infty$$

as $n \to \infty$.

Of course the second half of Theorem 4 follows from the proof of Theorem 2 with the same function f.

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